Extending the Lambda Calculus to Express Randomized and Quantumized Algorithms

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Abstract

This paper introduces a formal metalanguage called the lambda-q calculus for the specification of quantum programming languages. This metalanguage is an extension of the lambda calculus, which provides a formal setting for the specification of classical programming languages. As an intermediary step, we introduce a formal metalanguage called the lambda-p calculus for the specification of programming languages that allow true random number generation. We demonstrate how selected randomized algorithms can be programmed directly in the lambda-p calculus. We also demonstrate how satisfiability can be efficiently solved in the lambda-q calculus.

1. Introduction

This paper presents three formal language calculi, in increasing order of generality. The first one, the λ -calculus, is an old calculus for expressing functions. It is the basis of the semantics for many functional programming languages, including Scheme [4]. The second one, the λ^p -calculus, is a new calculus introduced here for expressing randomized functions. Randomized functions, instead of having a unique output for each input, return a distribution of results from which we sample once. The third one, the λ^q -calculus, is a new calculus introduced here for

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expressing quantumized functions. Quantumized functions also return a distribution of results, called a *superposition*, from which we sample once, but λ^q -terms have signs, and identical terms with opposite signs are removed before sampling from the result. Thus, superpositions can appear to shrink in size whereas distributions cannot. The λ^p -calculus is an extension of the λ -calculus. The λ^q -calculus is the most general but it is best presented in reference to the intermediary λ^p -calculus.

Although much research has been done on the hardware of quantum computation (c.f. [5], [6], [9]), none has focused on formalizing the software. Quantum Turing machines [5] have been introduced but there has been no quantum analogue to Church's λ -calculus. The λ -calculus has served as the basis for most programming languages since it was introduced by Alonzo Church [3] in 1936. It and other calculi make the implicit assumption that a term may be innocuously observed at any point. Such an assumption is hard to separate from a system of rewriting rules because to rewrite a term, you must have read it. However, as has been pointed out by Deutsch [7], any physical system is a computer. We may prepare it in some state, let it evolve according to its dynamics, and observe it periodically. Here, the notion of observation is crucial. One of the goals of these calculi is to make observation explicit in the formalism itself.

The intension of the λ^p - and the λ^q -calculi is to formalize computation on the level of *potentia* discussed by Heisenberg [8]. Heisenberg's quantum reality is a two-world model. One world is the world of *potentia*, events that haven't happened but could. The other world is the world of actual events that have occured and been observed. A goal of these calculi is to allow easy expression of algorithms that exist and operate in the world of *potentia* yet are, at conclusion, observed.

To this end, collections (distributions and superpositions) should be thought of with the following intuition. A collection is a bunch of terms that do not communicate with each other. When the collection is observed, at most one term in each collection will be the result of the observation. In the λ^q -calculus, the terms in a collection have signs, but still do not communicate with each other. The observation process somehow removes oppositely-signed terms. The key point is that in neither calculus can one write a term that can determine if it is part of a collection, how big the collection is, or even if its argument is part of a collection. Collections can be thought of as specifications of parallel terms whose execution does not depend on the execution of other terms in the same collection.

2. The Lambda Calculus

This section is a review of the λ -calculus and a reference for later calculi.

The λ -calculus is a calculus of functions. Any computable single-argument function can be expressed in the λ -calculus. Any computable multiple-argument function can be expressed in terms of computable single-argument functions. The λ -calculus is useful for encoding functions of arbitrary arity that return at most one output for each input. In particular, the λ -calculus can be used to express any (computable) algorithm. The definition of algorithm is usually taken to be Turing-computable.

2.1. Syntax

The following grammar specifies the syntax of the λ -calculus.

M	$\in Variable$ $\in LambdaTerm$ $\in Wff$	Variables Terms of the λ -calculus Well-formed formulas of the λ -calculus	
M	$ \begin{array}{ccc} ::= & x \\ & & M_1 M_2 \\ & & \lambda x.M \end{array} $	variable application abstraction	(2.1)
w	$::= M_1 = M_2$	well-formed formula	

To be strict, the subscripts above should be removed (e.g., the rule for well-formed formulas should read w := M = M) because M_1 and M_2 are not defined. However, we will maintain this incorrect notation to emphasize that the terms need not be identical.

With this abuse of notation, we can easily read the preceding definition as: a λ -term is a variable, or an application of two terms, or the abstraction of a term by a variable. A well-formed formula of the λ -calculus is a λ -term followed by the equality sign followed by a second λ -term.

We also adopt some syntactic conventions. Most importantly, parentheses group subexpressions. Application is taken to be left associative so that the term MNP is correctly parenthesized as (MN) P and not as M (NP). The scope of an abstraction extends as far to the right as possible, for example up to a closing parenthesis, so that the term $\lambda x.xx$ is correctly parenthesized as $(\lambda x.xx)$ and not as $(\lambda x.x)$ x.

2.2. Substitution

We will want to substitute arbitrary λ -terms for variables. We define the substitution operator, notated M[N/x] and read "M with all free occurences of x replaced by N." The definition of the free and bound variables of a term are standard. The set of free variables of a term M is written FV(M). There are six rules of substitution, which we write for reference.

1.
$$x [N/x] \equiv N$$

2. $y [N/x] \equiv y$ for variables $y \not\equiv x$
3. $(PQ) [N/x] \equiv (P [N/x]) (Q [N/x])$
4. $(\lambda x.P) [N/x] \equiv \lambda x.P$
5. $(\lambda y.P) [N/x] \equiv \lambda y. (P [N/x])$ if $y \not\equiv x$ and $y \notin FV(N)$
6. $(\lambda y.P) [N/x] \equiv \lambda z. (P [z/y] [N/x])$ if $y \not\equiv x$ and $y \in FV(N)$
and $z \notin FV(P) \bigcup FV(N)$

This definition will be extended in both subsequent calculi.

2.3. Reduction

The concept of reduction seeks to formalize rewriting rules. Given a relation R between terms, we may define the one-step reduction relation, notated \rightarrow_R , that is the contextual closure of R. We may also define the reflexive, transitive closure of the one-step reduction relation, which we call R-reduction and notate \rightarrow_R , and the symmetric closure of R-reduction, called R-interconvertibility and notated $=_R$

The essential notion of reduction for the λ -calculus is called β -reduction. It is based on the β -relation, which is the formalization of function invocation.

$$\beta \triangleq \{ ((\lambda x.M) N, M [N/x]) \mid M, N \in LambdaTerm, x \in Variable \}$$
 (2.3)

There is also the α -relation that holds of terms that are identical up to a consistent renaming of variables.

$$\alpha \triangleq \{(\lambda x.M, \lambda y.M [y/x]) \mid M \in LambdaTerm, y \notin FV(M)\}$$
 (2.4)

We will use this only sparingly.

2.4. Evaluation Semantics

By imposing an evaluation order on the reduction system, we are providing meaning to the λ -terms. The evaluation order of a reduction system is sometimes called an operational semantics or an evaluation semantics for the calculus. The evaluation relation is typically denoted \rightsquigarrow .

We use call-by-value evaluation semantics. A *value* is the result produced by the evaluation semantics. Call-by-value semantics means that the body of an abstraction is not reduced but arguments are evaluated before being passed into abstractions.

There are two rules for the call-by-value evaluation semantics of the λ -calculus.

$$\frac{1}{v \leadsto v} (\text{Refl}) \qquad \text{(for } v \text{ a value)}$$

$$\frac{M \leadsto \lambda x.P \quad N \leadsto N' \quad P\left[N'/x\right] \leadsto v}{MN \leadsto v} (\text{Eval})$$

2.5. Reference Terms

The following λ -terms are standard and are provided as reference for later examples.

Numbers are represented as Church numerals.

$$\underline{0} \equiv \lambda x.\lambda y.y \tag{2.5}$$

$$n \equiv \lambda x. \lambda y. x^n y \tag{2.6}$$

where the notation $x^n y$ means n right-associative applications of x onto y. It is abbreviatory for the term $\underbrace{x(x(\cdots(x\,y)))}_{n \text{ times}}$. When necessary, we can extend Church

numerals to represent both positive and negative numbers. For the remainder of the terms, we will not provide definitions. The predecessor of Church numerals is written \underline{P} . The successor is written \underline{S} .

The conditional is written \underline{IF} . If its first argument is truth, written \underline{T} , then it returns its second argument. If its first argument is falsity, written \underline{F} , then it returns its third argument. A typical predicate is $\underline{0?}$ which returns \underline{T} if its argument is the Church numeral $\underline{0}$ and \underline{F} if it is some other Church numeral.

The fixed-point combinator is written \underline{Y} . The primitive recursive function-building term is written \underline{PRIM} -REC and it works as follows. If the value of a function f at input n can be expressed in terms of n-1 and f(n-1), then that

function f is primitive recursive, and it can be generated by providing <u>PRIM-REC</u> with the function that takes the inputs n-1 and f(n-1) to produce f(n) and with the value of f at input 0. For example, the predecessor function for Church numerals can be represented as $\underline{P} \equiv \underline{PRIM-REC}(\lambda x. \lambda y. x) \underline{0}$.

3. The Lambda-P Calculus

The λ^p -calculus is an extension of the λ -calculus that permits the expression of randomized algorithms. In contrast with a computable algorithm which returns at most one output for each input, a randomized algorithm returns a distribution of answers from which we sample. There are several advantages to randomized algorithms.

- 1. Randomized algorithms can provide truly random number generators instead of relying on pseudo-random number generators that work only because the underlying pattern is difficult to determine.
- 2. Because they can appear to generate random numbers arbitrarily, randomized algorithms can model random processes.
- 3. Given a problem of finding a suitable solution from a set of possibilities, a randomized algorithm can exhibit the effect of choosing random elements and testing them. Such algorithms can sometimes have an *expected* running time which is considerably shorter than the running time of the computable algorithm that tries every possibility until it finds a solution.

3.1. Syntax

The following grammar describes the λ^p -calculus.

x	$\in Variable$	Variables	
M	$\in LambdaPTerm$	Terms of the λ^p -calculus	
w	$\in \mathit{WffP}$	Well-formed formulas of the λ^p -calculus	
M	$ \begin{array}{ll} ::= & x \\ & M_1 M_2 \\ & \lambda x.M \\ & M_1, M_2 \end{array} $	variable application abstraction collection	(3.1)
w	$::= M_1 = M_2$	well-formed formula	

Note that this grammar differs from the λ -calculus only in the addition of the fourth rule for terms. Therefore, all λ -terms can be viewed as λ^p -terms.

A λ^p -term is a variable, or an application of two terms, or the abstraction of a term by a variable, or a collection of two terms. It follows that a term may be a collection of a term and another collection, so that a term may actually have many nested collections.

We adhere to the same parenthesization and precedence rules as the λ -calculus with the following addition: collection is of lowest precedence and the comma is right associative. This means that the expression $\lambda x.x, z, y$ is correctly parenthesized as $(\lambda x.x)$, (z, y).

We introduce abbreviatory notation for collections. Let us write $[M_i^{i \in S}]$ for the collection of terms M_i for all i in the finite, ordered set S of natural numbers. We will write a..b for the ordered set $(a, a+1, \ldots, b)$. In particular, $[M_i^{i \in 1..n}]$ represents M_1, M_2, \ldots, M_n and $[M_i^{i \in n..1}]$ represents $M_n, M_{n-1}, \ldots, M_1$. More generally, let us allow multiple iterators in arbitrary contexts. Then, for instance,

$$\left[\lambda x. M_i^{i \in 1..n}\right] \equiv \lambda x. M_1, \lambda x. M_2, \dots, \lambda x. M_n$$

and

$$[M_i^{i \in 1..m} N_j^{j \in 1..n}] \equiv \begin{pmatrix} M_1 N_1, M_1 N_2, \dots, M_1 N_n, \\ M_2 N_1, M_2 N_2, \dots, M_2 N_n, \\ \vdots \\ M_m N_1, M_m N_2, \dots, M_m N_n \end{pmatrix}$$

Note that $[\lambda x. M_i^{i\in 1..n}]$ and $\lambda x. [M_i^{i\in 1..n}]$ are not the same term. The former is a collection of abstractions while the latter is an abstraction with a collection in its body. Finally, we allow this notation to hold of non-collection terms as well by identifying $[M_i^{i\in 1..1}]$ with M_1 even if M_1 is not a collection. To avoid confusion, it is important to understand that although this "collection" notation can be used for non-collections, we do not extend the definition of the word collection. A collection is still the syntactic structure defined in grammar (3.1).

With these additions, every term can be written in this bracket form. In particular, we can write a collection as $\left[\left[M_i^{i\in S_i}\right]_j^{j\in S}\right]$, or a collection of collections. Unfortunately, collections can be written in a variety of ways with this notation. The term M,N,P can be written as $\left[M_i^{i\in 1..3}\right]$ if $M_1\equiv M$ and $M_2\equiv N$ and $M_3\equiv P$; as $\left[M_i^{i\in 1..2}\right]$ if $M_1\equiv M$ and $M_2\equiv N,P$; or as $\left[M_i^{i\in 1..1}\right]$ if $M_1\equiv M,N,P$. However, it cannot be written as $\left[M_i^{i\in 1..4}\right]$ for any identification of the M_i . This observation inspires the following definition.

Definition 3.1. The cardinality of a term M, notated |M|, is that number k for which $[M_i^{i\in 1...k}] \equiv M$ for some identification of the M_i but $[M_i^{i\in 1..(k+1)}] \not\equiv M$ for any identification of the M_i .

Note that the cardinality of a term is always strictly positive.

3.2. Alternative Syntax

We present an alternative syntax for the λ^p -calculus that is provably equivalent to the one given above under certain assumptions. We will call the temporary calculus whose syntax we define below the $\lambda^{p'}$ -calculus to distinguish it from the one we will ultimately adopt.

The following grammar describes the syntax of the $\lambda^{p'}$ -calculus.

x	$\in Variable$	Variables	
M	$\in LambdaTerm$	Terms of the λ -calculus	
C	$\in LambdaP'Term$	Terms of the $\lambda^{p'}$ -calculus	
w	$\in \mathit{WffP'}$	Well-formed formulas of the $\lambda^{p'}$ -calculus	
M	$ \begin{array}{ll} ::= & x \\ & M_1 M_2 \\ & \lambda x. M \end{array} $	variable application abstraction	(3.2)
C	$ \begin{array}{ll} ::= & M \\ & C_1, C_2 \\ & C_1 C_2 \end{array} $	term construction collection application	
w	$::= C_1 = C_2$	well-formed formula	

The syntax of the λ^p -calculus, grammar (3.2), allows the same terms and well-formed formulas as the syntax of the $\lambda^{p'}$ -calculus, grammar (3.1), if we identify abstractions of collections with the appropriate collection of abstractions, and applications of collections with collections of applications.

Theorem 3.2. If we have, in the λ^p -calculus, that

$$\left[\lambda x. M_i^{i \in 1..n}\right] \equiv \lambda x. \left[M_i^{i \in 1..n}\right] \tag{3.3}$$

and

$$\left[M_i^{i \in 1..m} N_j^{j \in 1..n}\right] \equiv \left[M_i^{i \in 1..m}\right] \left[N_j^{j \in 1..n}\right] \tag{3.4}$$

then w is a well-formed formula in the λ^p -calculus if and only if it is a well-formed formula in the $\lambda^{p'}$ -calculus.

Proof. It is sufficient to show that an arbitrary $\lambda^{p'}$ -term is a λ^p -term and vice versa.

First we show that an arbitrary $\lambda^{p'}$ -term is a λ^p -term by structural induction. From (3.2), a $\lambda^{p'}$ -term C is either a λ -term, a construction, or a collection application. If C is a λ -term, then it is a λ^p -term. If it is a construction C_1, C_2 , then C is a λ^p -collection term C_1, C_2 because C_1 and C_2 are λ^p -terms by the induction hypothesis. Finally, if C is a collection application C_1C_2 , then, since C_1 and C_2 are λ^p -terms by the induction hypothesis and C_1C_2 is a λ^p -application term, C is a λ^p -term.

Now we show that an arbitrary λ^p -term is a $\lambda^{p'}$ -term by structural induction. If M is a λ^p -term, it is either a variable, an application, a collection, or an abstraction. If M is a variable, then it is a λ -term and therefore a $\lambda^{p'}$ -term. If M is an application PQ, then by the induction hypothesis P and Q are $\lambda^{p'}$ -terms, so that PQ is a $\lambda^{p'}$ -collection application and M is a $\lambda^{p'}$ -term. If M is a collection, then by the induction hypothesis it is a collection of λ^p -terms that are $\lambda^{p'}$ -terms, so that M is also a $\lambda^{p'}$ -construction.

If M is an abstraction $\lambda x.N$, then by the induction hypothesis, N is a $\lambda^{p'}$ -term. Therefore, N is either a λ -term, a construction, or a collection application. If N is a λ -term, then M is a $\lambda^{p'}$ -term. If N is a $\lambda^{p'}$ -construction, then it is also a λ^p -collection term by the first part of this proof. Therefore, M is an abstraction over a collection, and by assumption (3.3) is identical to a collection over abstractions. By the induction hypothesis, each of the abstractions in the collection is a $\lambda^{p'}$ -term, so the collection itself is a $\lambda^{p'}$ -construction. Therefore, M is a $\lambda^{p'}$ -term. Finally, if N is a collection application, then by assumption (3.4) it is identical to a collection of applications. Therefore, N is a λ^p -collection term. By the same reasoning as in the previous case, it follows that M is a $\lambda^{p'}$ -term.

This completes the proof. ■

The λ^p -calculus seems more expressive than the $\lambda^{p'}$ -calculus because it allows terms to be collections of other terms. We have seen that with the two assumptions (3.3) and (3.4), the two calculi are equally expressive. Without these assumptions, some abstractions can be expressed in the λ^p -calculus that cannot be expressed in the $\lambda^{p'}$ -calculus. Are these assumptions justifiable?

The first assumption (3.3) states that the abstraction of a collection is syntactically identical to the collection of the abstractions. For example, the term $\lambda x.(x,xx)$ is claimed to be identical to the term $\lambda x.x, \lambda x.xx$. Given our intuitive understanding of what these terms represent, it is indisputable that these two terms are equal in a semantic sense. Applying each term to arbitrary inputs ought to yield statistically indistinguishable results. However, the question is: should we identify these terms on a syntactic level? Certainly the two λ -terms $\lambda x.x$ and $\lambda x.(\lambda y.y)x$ are semantically equivalent, but we do not identify them on a syntactic level.

The other assumption (3.4) states that the application of two collections is the collection of all possible applications of terms in the two collections. For example, the term (M,N) (P,Q) is claimed to be identical to the term MP,MQ,NP,NQ. Again, these two would, given our intuition, be equal in the statistical sense, but should they represent the same syntactic structure? Even if the two terms represent the same thing in the real world, that is, if they share the same denotation, it does not follow that they should be syntactically identical. β -reduction preserves denotation, but we do not say that a term and what it reduces to are syntactically identical.

On the other hand, we do want to identify some terms that are written differently. For example, the order of terms in a collection ought not distinguish terms. The terms M, N and N, M should be identified, given our intuitive understanding of what these terms mean. Identifying unordered terms is not uncommon and is done in other calculi [1]. How do we decide whether to identify certain pairs of terms or to define a notion of reduction for them?

We want to identify terms when the differences do not affect computation and result from the limiting nature of writing. Identifying collections with different orders is a workaround for the sequentiality and specificity of the comma operator. Together with the grammar, such an identification clarifies the terms of discourse. However, this reasoning does not apply to the assumptions (3.3) and (3.4) because these assumptions are trying to identify terms that bear only a semantic relationship to each other.

Much as we refrain from identifying a term with its β -reduced form, we do not want to identify an application (abstraction) of collections with a collection of applications (abstractions). We may choose instead to capture this relationship in the form of a relation and associated reductions. Such is the approach we will adopt here.

Of the two grammars, we choose the λ^p -calculus because it is the more gen-

eral one. We will define the γ -relation to hold of an application of collections and a collection of applications. However, we will neither identify nor provide a relation for the analogous abstraction relationship of the pair of terms identified in assumption (3.3), because such a step would be redudant. By the observation function we will define in §3.6, observing an abstraction of collections is tantamount to observing a collection of abstractions, so no new power or expressibility would be gained.

3.3. Syntactic Identities

We define substitution of terms in the λ^p -calculus as an extension of substitution of terms in the λ -calculus. In addition to the six rules of the λ -calculus, we introduce one for collections.

$$(P,Q)[N/x] \equiv (P[N/x], Q[N/x])$$
 (3.5)

We identify terms that are collections but with a possibly different ordering. We also identify nested collections with the top-level collection. The motivation for this is the conception that a collection is an unordered set of terms. Therefore we will not draw a distinction between a set of terms and a set of a set of terms.

We adopt the following axiomatic judgement rules.

$$\frac{\overline{M, N \equiv N, M}(\text{ClnOrd})}{(M, N), P \equiv M, (N, P)}(\text{ClnNest})$$

With these axioms, ordering and nesting become innocuous. As an example here is the proof that $A, (B, C), D \equiv A, C, B, D$. For clarity, we parenthesize fully and underline the affected term in each step.

$$\begin{array}{cccc} \underline{A,((B,C),D)} & \equiv & ((\underline{B,C}),D),A & (\text{ClnOrd}) \\ & \equiv & ((\underline{C,B}),D),A & (\text{ClnOrd}) \\ & \equiv & \underline{(C,(B,D))},A & (\text{ClnNest}) \\ & \equiv & \overline{A,(C,(B,D))} & (\text{ClnOrd}) \end{array}$$

We now show that ordering and parenthesization are irrelevant in general.

Theorem 3.3. If the *n* ordered sets S_i , $1 \leq i \leq n$, are distinct and Π is a permutation of the ordered set 1..n, then $\left[\left[M_i^{i \in S_i}\right]_j^{j \in 1..n}\right] \equiv \left[M_i^{i \in S_1 S_2 \cdots S_n}\right]$ and $\left[M_i^{i \in 1..n}\right] \equiv \left[M_i^{i \in \Pi}\right]$, where juxtaposition of ordered sets denotes extension (e.g., (1..3)(5..7) = (1, 2, 3, 5, 6, 7)).

Proof. We prove this theorem by induction on n.

There are two base cases. When n = 1, the claim holds trivially. When n = 2, the claim follows from the (ClnOrd) axiom.

For the inductive case, we consider $M \equiv \left[M_i^{i \in 1..(n+1)} \right] \equiv M_1, \left[M_i^{i \in 2..(n+1)} \right]$ and assume the claim holds for all collections of n or fewer terms, and that $n \geq 2$.

To show parenthesization invariance, we write $M \equiv P, Q$ where $P \equiv \begin{bmatrix} P_i^{i \in 1..n} \end{bmatrix}$ and Q are collections. It will be sufficient to show that $M \equiv \begin{bmatrix} P_i^{i \in 1..(n-1)} \end{bmatrix}, (P_n, Q)$. By the induction hypothesis, we may parenthesize P arbitrarily. We choose to parenthesize P left-associatively as

$$(((P_1, P_2), P_3 \cdots P_{n-2}), P_{n-1}), P_n.$$

Then, by the (ClnNest) axiom, $M \equiv (((P_1, P_2), P_3 \cdots P_{n-2}), P_{n-1}), (P_n, Q)$, which is identical to $\left[P_i^{i \in 1...(n-1)}\right], (P_n, Q)$ by the reordering allowed by the induction hypothesis. This completes the proof of parenthesization invariance.

To show reordering invariance, note that the permutation Π of 1.. (n+1) either has 1 as its first element or it does not. If it does, then by the induction hypothesis, $\left[M_i^{i\in 2..(n+1)}\right]$ can be reordered in an arbitrary manner, so that $M\equiv \left[M_i^{i\in \Pi}\right]$. If it does not, then the first element of Π is an integer k between 2 and n-1. By the induction hypothesis, we can reorder $\left[M_i^{i\in 2..(n+1)}\right]$ as M_k , $\left[M_i^{i\in 2..(n+1)-\{k\}}\right]$, where we write 2..(n+1)-k for the ordered set $(2,3,\ldots,k-1,k+1,\ldots,n+1)$. Then, underlying the affected term, we get

$$\begin{split} M &\equiv M_1, M_k, \left[M_i^{i \in 2..(n+1) - \{k\}}\right] & \text{by reordering} \\ &\equiv \left(\underline{M_1, M_k}\right), \left[M_i^{i \in 2..(n+1) - \{k\}}\right] & \text{by the (ClnNest) axiom} \\ &\equiv \left(M_k, M_1\right), \left[M_i^{i \in 2..(n+1) - \{k\}}\right] & \text{by the (ClnOrd) axiom} \\ &\equiv \overline{M_k, M_1, \left[M_i^{i \in 2..(n+1) - \{k\}}\right]} & \text{by the (ClnNest) axiom} \end{split}$$

Then, by the induction hypothesis, M_1 , $\left[M_i^{i \in 2..(n+1)-\{k\}}\right]$ can be reordered arbitrarily so that M can be reordered to fit the permutation Π . This completes the proof of reordering invariance and of the theorem.

Aside, it no longer matters that we took the comma to be right associative since, with these rules, any arbitrary parenthesization of a collection does not change the syntactic structure.

Because of this theorem, we can alter the abbreviatory notation and allow arbitrary unordered sets in the exponent. This allows us to write, for instance, $\left[M_i^{i\in 1..n-\{j\}}\right] \equiv M_1, M_2, \ldots, M_{j-1}, M_{j+1}, \ldots, M_n$ where a..b is henceforth taken to be the unordered set $\{a, a+1, \ldots, b\}$ and the subtraction in the exponent represents set difference.

This also subtly alters the definition of *cardinality* (3.1). Whereas before the cardinality of a term like (x, y), z was 2, because of this theorem, it is now 3. Because every λ^p -term is finite, the cardinality is well-defined.

3.4. Reductions

The relation of collection application is called the γ -relation. It holds of a term that is an application at least one of whose operator or operand is a collection, and the term that is the collection of all possible pairs of applications.

$$\gamma^{p} \triangleq \left\{ \begin{array}{l} \left(\left[M_{i}^{i \in 1..m} \right] \left[N_{j}^{j \in 1..n} \right], \left[M_{i}^{i \in 1..m} N_{j}^{j \in 1..n} \right] \right) \\ \text{such that } M_{i}, N_{j} \in LambdaPTerm, \ m > 1 \text{ or } n > 1 \end{array} \right\}$$
 (3.6)

The γ -relation is our solution to the concerns of §3.2 regarding claim (3.4). We will omit the superscript except to disambiguate from the γ -relation of the λ^q -calculus.

We generate the reduction relations as described in §2.3 to get the relations of γ -reduction in one step \rightarrow_{γ} , γ -reduction $\twoheadrightarrow_{\gamma}$, and γ -interconvertibility $=_{\gamma}$.

The γ -relation is Church-Rosser.

Theorem 3.4. For λ^p -terms M, R, S, if $M \twoheadrightarrow_{\gamma} R$ and $M \twoheadrightarrow_{\gamma} S$ then there exists a λ^p -term T such that $R \twoheadrightarrow_{\gamma} T$ and $S \twoheadrightarrow_{\gamma} T$.

This is shown in the standard way by proving the associated strip lemma.

As a result of this theorem, γ -normal forms, when they exist, are unique. We now show that all terms have γ -normal forms.

Theorem 3.5. For every λ^p -term M there exists another λ^p -term N such that $M \to_{\gamma} N$ and N has no γ -redexes.

Proof. The proof is by structural induction on M.

If $M \equiv x$ is a variable, there are no γ -redexes, so $N \equiv M$.

If $M \equiv \lambda x.P$ is an abstraction, then the only γ -redexes, if any, are in P. By the induction hypothesis, there exists a term P' such that $P \twoheadrightarrow_{\gamma} P'$ and P' has no γ -redexes. Then $M \twoheadrightarrow_{\gamma} \lambda x.P' \equiv N$ and N has no γ -redexes either.

If $M \equiv PQ$ is an application, then there exist terms P', Q' such that $P \twoheadrightarrow_{\gamma} P'$ and $Q \twoheadrightarrow_{\gamma} Q'$ and neither P' nor Q' have γ -redexes. Let $\left[P_i^{i \in 1..|P'|}\right] \equiv P'$ and $\left[Q_i^{i \in 1..|Q'|}\right] \equiv Q'$. Then $M \twoheadrightarrow_{\gamma} P'Q' \to_{\gamma} \left[P_i^{i \in 1..|P'|}Q_j^{j \in 1..|Q'|}\right] \equiv N$ where none of the P_i or Q_j are collections, by the definition of cardinality. Also, since neither P' nor Q' had γ -redexes, none of the P_i or Q_j have γ -redexes either. Therefore, N does not have any γ -redexes.

N does not have any γ -redexes. If $M \equiv \left[M_i^{i \in 1..|M|} \right]$ is a collection, then for each M_i there exists a term N_i such that $M_i \twoheadrightarrow_{\gamma} N_i$ and N_i has no γ -redexes. Then $M \twoheadrightarrow_{\gamma} \left[N_i^{i \in 1..|M|} \right] \equiv N$ and since none of the N_i have γ -redexes, neither does N.

This exhausts the cases and completes the proof. ■

Therefore, all λ^p -terms have normal forms and they are unique, so we may write $\gamma(M)$ for the γ -normal form of a λ^p -term M.

We extend the β -relation to apply to collections.

$$\beta^{p} \triangleq \left\{ \begin{array}{l} \left((\lambda x.M) \left[N_{i}^{i \in S} \right], \left[M \left[N_{i}^{i \in S} / x \right] \right] \right) \\ \text{such that } M \text{ and } \left[N_{i}^{i \in S} \right] \in LambdaPTerm, \ x \in Variable \end{array} \right\}$$
 (3.7)

where $[M[N_i^{i \in S}/x]]$ is the collection of terms M with N_i substituted for free occurrences of x in M, for $i \in S$.

One-step β -reduction in the λ^p -calculus \to_{β^p} is different from that of the λ -calculus because the grammar is extended. (Note that we omit the superscript on β -reduction when there is no ambiguity about which calculus is under consideration.) Therefore, we need to prove that the β -relation is still Church-Rosser in the λ^p -calculus. This is easy to do but not helpful because the appropriate notion of reduction in the λ^p -calculus is not β -reduction, but β -reduction with γ -reduction to normal form after each β -reduction step.

We define the relation $\beta\gamma$ which is just like the β -relation except that the resultant term is in γ -normal form.

$$\beta \gamma \triangleq \{ (M, \gamma(N)) \mid (M, N) \in \beta^p \}$$
 (3.8)

The $\beta\gamma\text{-relation}$ is Church-Rosser.

Theorem 3.6. For λ^p -terms M, R, S, if $M \to_{\beta} R$ and $M \to_{\beta} S$ then there exists a λ^p -term T such that $R \to_{\beta} T$ and $S \to_{\beta} T$.

Again the proof follows the standard framework.

3.5. Evaluation Semantics

We extend the call-by-value evaluation semantics of the λ -calculus.

There are three rules for the call-by-value evaluation semantics of the λ^p -calculus. We modify the definition of a value v to enforce that v has no γ -redexes.

$$\frac{\overline{v \leadsto v}(\text{Refl}) \qquad \text{(for } v \text{ a value)}}{\gamma\left(M\right) \leadsto \lambda x.P \quad \gamma\left(N\right) \leadsto N' \quad \gamma\left(P\left[N'/x\right]\right) \leadsto v} (\text{Eval})$$

$$\frac{\gamma\left(M\right) \leadsto v_1 \quad \gamma\left(N\right) \leadsto v_2}{(M,N) \leadsto (v_1,v_2)} (\text{Coll})$$

3.6. Observation

We define an observation function Θ from λ^p -terms to λ -terms. We employ the random number generator RAND, which samples one number from a given set of numbers.

$$\Theta\left(x\right) = x \tag{3.9}$$

$$\Theta(\lambda x.M) = \lambda x.\Theta(M) \tag{3.10}$$

$$\Theta(M_1 M_2) = \Theta(M_1) \Theta(M_2) \tag{3.11}$$

$$\Theta\left(M \equiv \left[M_i^{i \in 1..|M|}\right]\right) = M_{RAND(1..|M|)} \tag{3.12}$$

The function Θ is total because every λ^p -term is mapped to a λ -term. Note that for an arbitrary term T we may write $\Theta(T) = T_{RAND(S)}$ for some possibly singleton set of natural numbers S and some collection of terms $T_i^{i \in S}$.

Definition 3.7. We say that $\Theta(M) = M_{RAND(S)}$ is statistically indistinguishable from $\Theta(N) = N_{RAND(S')}$, written $\Theta(M) \stackrel{d}{=} \Theta(N)$, if there exist positive integers m and n and a total, surjective mapping φ between S and S' such that, for each $k \in S$, $M_k \equiv N_{\varphi(k)}$ and

$$\frac{\left| \left[M_i^{i \in \{j \mid M_j \equiv M_k\}} \right] \right|}{|S|} = \frac{\left| \left[N_i^{i \in \{\varphi(j) \mid M_j \equiv M_k\}} \right] \right|}{|S'|}$$
(3.13)

that is, if the proportion of terms in M identical to M_k is the same as the proportion of terms in N identical to $N_{\varphi(k)}$, for all $k \in S$. In particular, if the mapping φ is an isomorphism, $\Theta(M)$ is said to be statistically identical to $\Theta(N)$, written $\Theta(M) \stackrel{D}{=} \Theta(N)$.

Because the mapping φ is total and surjective, statistical indistinguishability is a symmetric property. Note that statististical identity is a restatement of the theorem of parenthesization and ordering invariance (3.3). We now show that observing a λ^p -term is statistically indistinguishable from observing its γ -normal form.

Theorem 3.8. If
$$M \rightarrow_{\gamma} N$$
 then $\Theta(M) \stackrel{d}{=} \Theta(N)$.

Proof. The proof is by structural induction on M. The induction hypothesis is stronger than required and states that if $M \to_{\gamma} N$ then $\Theta(M)$ and $\Theta(N)$ are statistically identical.

If M is a variable then $M \equiv N$ and $\Theta\left(M\right) = \Theta\left(N\right)$. In particular, $\Theta\left(M\right) \stackrel{D}{=} \Theta\left(N\right)$.

If $M \equiv \lambda x.P$ is an abstraction then $N \equiv \lambda x.P'$ where $P \twoheadrightarrow_{\gamma} P'$. By the induction hypothesis, $\Theta(P) \stackrel{D}{=} \Theta(P')$, and by definition $\Theta(M) = \lambda x.\Theta(P) \stackrel{D}{=} \lambda x.\Theta(P') = \Theta(\lambda x.P') = \Theta(N)$.

If $M \equiv PQ$ is an application, then either N is an application or a collection. If N is an application, then $N \equiv P'Q'$ and, since $P \twoheadrightarrow_{\gamma} P'$ and $Q \twoheadrightarrow_{\gamma} Q'$, we have by the induction hypothesis that $\Theta(P) \stackrel{D}{=} \Theta(P')$ with isomorphism φ_P and $\Theta(Q) \stackrel{D}{=} \Theta(Q')$ with isomorphism φ_Q . We write

$$\Theta(P) = P_{RAND(S_P)}
\Theta(P') = P'_{RAND(S_{P'})}
\Theta(Q) = Q_{RAND(S_Q)}
\Theta(Q') = Q'_{RAND(S_{Q'})}
\Theta(M) = M_{RAND(S_M)}
\Theta(N) = N_{RAND(S_N)}$$

and note that it is sufficient to exhibit an isomorphism between $\Theta(M) = \Theta(PQ)$ and $\Theta(N) = \Theta(P'Q')$. Without loss of generality, let $S_p = S_{p'} = 1..p$ and

 $S_q = S_{q'} = 1..q$ so we can write, for $i \in 1...(pq)$,

$$M_i \equiv P_{1+(i-1)\backslash q}Q_{1+(i-1)\bmod q}$$
 (3.14)

$$N_i \equiv P'_{1+(i-1)\backslash q} Q'_{1+(i-1) \bmod q} \tag{3.15}$$

where $x \setminus y = \left\lfloor \frac{x}{y} \right\rfloor$. Remember that by theorem (3.3), order and parenthesization does not matter. Given isomorphisms φ_P and φ_Q , we need to find an isomorphism, φ , between the M_i and N_i . We rewrite

$$M_i \equiv P'_{\varphi_p(1+(i-1)\setminus q)}Q_{1+(i-1)\bmod q}$$
 (3.16)

$$\equiv P'_{\varphi_{P}(1+(i-1)\backslash q)}Q'_{\varphi_{Q}(1+(i-1)\bmod q)} \tag{3.17}$$

$$\equiv N_{1+(\varphi_P(1+(i-1)\backslash q)-1)q+(\varphi_Q(1+(i-1)\bmod q)-1)}$$
(3.18)

where the last identity follows from rewriting identity (3.15) as

$$N_{1+(j-1)q+(k-1)} \equiv P_j Q_k \tag{3.19}$$

where $j \in 1...p$, $k \in 1...q$. Thus, the isomorphism $\varphi(i) = 1 + (\varphi_P(1 + (i-1) \setminus q) - 1) q + (\varphi_Q(1 + (i-1) \mod q) - 1)$ satisfies the definition of statistical identity (3.7) so $\Theta(M) \stackrel{D}{=} \Theta(N)$.

Finally, if $M \equiv \left[M_i^{i\in 1..|M|}\right]$ is a collection, then $N \equiv \left[N_i^{i\in 1..|M|}\right]$ must be a collection, too, where each $M_i \twoheadrightarrow_{\gamma} N_i$. By the induction hypothesis, $\Theta\left(M_i\right) \stackrel{D}{=} \Theta\left(N_i\right)$ for each $i \in 1..|M|$. In particular, for each M_i there is an N_j such that $M_i \equiv N_j$. The isomorphism follows by identifying each i with the appropriate j.

This exhausts the cases and completes the proof.

3.7. Observational Semantics

We provide another type of semantics for the λ^p -calculus called its *observational* semantics. A formalism's observational semantics expresses the computation as a whole: preparing the input, waiting for the evaluation, and observing the result. The observational semantics relation between λ^p -terms and λ -terms is denoted \multimap . It is given by a single rule for the λ^p -calculus.

$$\frac{M \leadsto v \quad \Theta(v) = N}{M \multimap N} \text{(ObsP)}$$
(3.20)

3.8. Examples

A useful term of the λ^p -calculus is a random number generator. We would like to define a term that takes as input a Church numeral \underline{n} and computes a collection of numerals from $\underline{0}$ to \underline{n} . This can be represented by the following primitive recursive λ^p -term.

$$\underline{\mathbf{R}} \equiv \underline{\mathbf{PRIM}} - \underline{\mathbf{REC}} (\lambda k. \lambda p. (k, p)) \underline{\mathbf{0}}$$
(3.21)

Then for instance $\underline{R3} = (3, 2, 1, 0)$.

The following term represents a random walk. Imagine a man that at each moment can either walk forward one step or backwards one step. If he starts at the point 0, after n steps, what is the distribution of his position?

$$\underline{\mathbf{W}} \equiv \underline{\mathbf{PRIM}} - \underline{\mathbf{REC}} \left(\lambda k. \lambda p. \left(\underline{\mathbf{P}} p, \underline{\mathbf{S}} p \right) \right) \underline{\mathbf{0}}$$
 (3.22)

We assume we have extended Church numerals to negative numbers as well. This can be easily done by encoding it is a pair. We will show some of the highlights of the evaluation of $\underline{W}\underline{3}$. Note that $\underline{W}\underline{1} = (-1,\underline{1})$.

$$\underline{W3} = \underline{P}(\underline{W2}), \underline{S}(\underline{W2})
= \underline{P}(\underline{P}(\underline{W1}), \underline{S}(\underline{W1})), \underline{S}(\underline{P}(\underline{W1}), \underline{S}(\underline{W1}))
= \underline{P}(\underline{P}(\underline{W1}), \underline{S}(\underline{W1})), \underline{S}(\underline{P}(\underline{W1}), \underline{S}(\underline{W1}))
= \underline{P}(\underline{P}(\underline{-1}, \underline{1}), \underline{S}(\underline{-1}, \underline{1})), \underline{S}(\underline{P}(\underline{-1}, \underline{1}), \underline{S}(\underline{-1}, \underline{1}))
= \underline{P}((\underline{-2}, \underline{0}), (\underline{0}, \underline{2})), \underline{S}((\underline{-2}, \underline{0}), (\underline{0}, \underline{2}))
= ((\underline{-3}, \underline{-1}), (\underline{-1}, \underline{1})), ((\underline{-1}, \underline{1}), (\underline{1}, \underline{3}))
\equiv (-3, -1, -1, \underline{1}, -1, \underline{1}, \underline{1}, \underline{3})$$
(3.23)

Observing <u>W 3</u> yields $\underline{-1}$ with probability $\frac{3}{8}$, $\underline{1}$ with probability $\frac{3}{8}$, $\underline{-3}$ with probability $\frac{1}{8}$, and $\underline{3}$ with probability $\frac{1}{8}$.

4. The Lambda-Q Calculus

The λ^q -calculus is an extension of the λ^p -calculus that allows easy expression of quantumized algorithms. A quantumized algorithm differs from a randomized algorithm in allowing negative probabilities and in the way we sample from the resulting distribution.

Variables and abstractions in the λ^q -calculus have *phase*. The phase is nothing more than a plus or minus sign, but since the result of a quantumized algorithm is a distribution of terms with phase, we call such a distribution by the special name

superposition. The major difference between a superposition and a distribution is the observation procedure. Before randomly picking an element, a superposition is transformed into a distribution by the following two-step process. First, all terms in the superposition that are identical except with opposite phase are cancelled. They are both simply removed from the superposition. Second, the phases are stripped to produce a distribution. Then, an element is chosen from the distribution randomly, as in the λ^p -calculus.

The words *phase* and *superposition* come from quantum physics. An electron is in a superposition if it can be in multiple possible states. Although the phases of the quantum states may be any angle from 0° to 360° , we only consider binary phases. Because we use solely binary phases, we will use the words *sign* and *phase* interchangeably in the sequel.

A major disadvantage of the λ^p -calculus is that it is impossible to compress a collection. Every reduction step at best keeps the collection the same size. Quantumized algorithms expressed in the λ^q -calculus, on the other hand, can do this as easily as randomized algorithms can generate random numbers. That is, λ^q -terms can contain subterms with opposite signs which will be removed during the observation process.

4.1. Syntax

The following grammar describes the λ^q -calculus.

S	$\in Sign$	Sign, or phase	
x	$\in Variable$	Variables	
M	$\in LambdaQTerm$	Terms of the λ^q -calculus	
w	$\in \mathit{WffQ}$	Well-formed formulas of the λ^q -calculus	
S	∷= + −	positive negative	(4.1)
M	$ \begin{array}{ccc} ::= & Sx \\ & M_1 M_2 \\ & S \lambda x. M \\ & M_1, M_2 \end{array} $	signed variable application signed abstraction collection	
w	$::= M_1 = M_2$	well-formed formula	

Terms of the λ^q -calculus differ from terms of the λ^p -calculus only in that

variables and abstractions are *signed*, that is, they are preceded by either a plus (+) or a minus (-) sign. Just as λ -terms could be read as λ^p -terms, we would like λ^p -terms to be readable as λ^q -terms. However, λ^p -terms are unsigned and cannot be recognized by this grammar.

Therefore, as is traditionally done with integers, we will omit the positive sign. An unsigned term in the λ^q -calculus is abbreviatory for the same term with a positive sign. With this convention, λ^p -terms can be seen as λ^q -terms all of whose signs are positive. Also, so as not to confuse a negative sign with subtraction, we will write it with a logical negation sign (\neg). With these two conventions, the λ^q -term $+\lambda x + x - x$ is written simply $\lambda x \cdot x \neg x$.

Instead of these conventions, we could just as well have rewritten the grammar of signs so that the positive sign was spelled with the empty string (traditionally denoted by the Greek letter ϵ) and the negative signs was spelled with the logical negation sign. We would have gotten the alternative grammar below.

$$S' ::= \epsilon \tag{4.2}$$

However, this would have suggested an asymmetry between positive and negative signs and allowed the interpretation that negatively signed terms are a "type" of positively signed terms. On the contrary, we want to emphasize that there are two distinct kinds of terms, positive and negative, and neither is better than the other. There is no good reason why λ^p -terms should be translated into positively signed λ^q -terms and not negatively signed ones. This arbitrariness is captured better as a convention than a definition.

Finally, we adhere to the same parenthesization and precedence rules as the λ^p -calculus. In particular, we continue the use of the abbreviatory notation $[M_i^{i \in S}]$ for collections of terms, although we will not recast the parenthesization and ordering invariance theorem (3.3) for terms of the λ^q -calculus. The modifications to the proof are mild.

4.2. Syntactic Identities

We want to give a name to the relationship between two terms that differ only in sign.

Definition 4.1. A λ^q -term M is the opposite of a λ^q -term N, written $M \equiv \overline{N}$, if either

$$M \equiv S_1 x$$
 and $N \equiv S_2 x$

where S_1 and S_2 are different signs, or

$$M \equiv S_1 \lambda x. M'$$
 and $N \equiv S_2 \lambda x. N'$

where S_1 and S_2 are different signs, and $M' \equiv N'$.

Note that not all terms have opposites but if $M \equiv \overline{N}$ then it follows that $N \equiv \overline{M}$. We define substitution of terms in the λ^q -calculus as a modification of substitution of terms.

tution of terms in the λ^{-} -calculus. We rewrite the seven rules of the λ^{p} -calculus to take account of the signs of the terms. First, we introduce the function notated by sign concatenation, defined by the following four rules:

$$++ \mapsto + \tag{4.3}$$

$$+- \mapsto - \tag{4.4}$$

$$-+ \mapsto - \tag{4.5}$$

$$-- \mapsto + \tag{4.6}$$

Notating this in our alternative syntax for signs (4.2), these rules can be summarized by the single rewrite rule

$$\neg \neg \mapsto \epsilon \tag{4.7}$$

because the concatenation of a sign S with ϵ is just S again. Now we can use this function in the following substitution rules.

- 1. $(Sx)[N/x] \equiv SN$ for variables $y \not\equiv x$
- 2. $(Sy)[N/x] \equiv Sy$
- 3. $(PQ)[N/x] \equiv (P[N/x])(Q[N/x])$
- 4. $(S\lambda x.P)[N/x] \equiv S\lambda x.P$

5.
$$(S\lambda y.P)[N/x] \equiv S\lambda y.(P[N/x])$$
 if $y \not\equiv x$ and $y \notin FV(N)$ (4.8)

6.
$$(S\lambda y.P)[N/x] \equiv S\lambda z. (P[z/y][N/x])$$
 if $y \not\equiv x$ and $y \in FV(N)$ and $z \notin FV(P) \bigcup FV(N)$

7.
$$(P,Q)[N/x] \equiv (P[N/x], Q[N/x])$$

Where did we use the sign concatenation function in the above substitution rules? It is hidden in rule (1). Consider $(\neg x)[\neg \lambda y.y/x] \equiv \neg \neg \lambda y.y$. This is not a λ^q -term by grammar (4.1). Applying the sign concatenation function yields $\lambda y.y$, which is a λ^q -term. However, we could not have rewritten rule (1) to be explicit about the sign of N because N may be an application or a collection and therefore not have a sign.

4.3. Reduction

The γ -relation of the λ^q -calculus is of the same form as that of the λ^p -calculus.

$$\gamma^{q} \triangleq \left\{ \begin{array}{l} \left(\left[M_{i}^{i \in 1..m} \right] \left[N_{j}^{j \in 1..n} \right], \left[M_{i}^{i \in 1..m} N_{j}^{j \in 1..n} \right] \right) \\ \text{such that } M_{i}, N_{j} \in LambdaQTerm, \ m > 1 \text{ or } n > 1 \end{array} \right\}$$

$$(4.9)$$

We omit the superscript when it is clear from context if the terms under consideration are λ^p -terms or λ^q -terms. We still write $\gamma(M)$ for the γ -normal form of M. Theorems (3.4) and (3.5) are easily extendible to terms of the λ^q -calculus so $\gamma(M)$ is well-defined.

We extend the β -relation to deal properly with signs.

$$\beta^{q} \triangleq \left\{ \begin{array}{l} ((S\lambda x.M) N, SM [N/x]) \\ \text{such that } S \in Sign, S\lambda x.M \text{ and } N \in LambdaQTerm \end{array} \right\}$$
 (4.10)

We refer to β -reduction for the λ -calculus, the λ^p -calculus, and the λ^q -calculus all with the same notation when there is no risk of ambiguity.

4.4. Evaluation Semantics

We modify the call-by-value evaluation semantics of the λ^p -calculus.

There are three rules for the call-by-value evaluation semantics of the λ^q -calculus.

$$\frac{1}{v \leadsto v}(\text{Refl}) \qquad \text{(for } v \text{ a value)}$$

$$\frac{\gamma(M) \leadsto S\lambda x.P \quad \gamma(N) \leadsto N' \quad \gamma(SP[N'/x]) \leadsto v}{MN \leadsto v}(\text{Eval})$$

$$\frac{\gamma(M) \leadsto v_1 \quad \gamma(N) \leadsto v_2}{(M,N) \leadsto (v_1,v_2)}(\text{Coll})$$

4.5. Observation

We define an observation function Ξ from λ^q -terms to λ -terms as the composition of a function Δ from λ^q -terms to λ^p -terms with the observation function Θ from λ^p -terms to λ -terms defined in (3.6). Thus, $\Xi = \Theta \circ \Delta$ where we define Δ as

follows.

$$\Delta(Sx) = x \tag{4.11}$$

$$\Delta (S\lambda x.M) = \lambda x.\Delta (M) \tag{4.12}$$

$$\Delta (M_1 M_2) = \Delta (M_1) \Delta (M_2) \tag{4.13}$$

$$\Delta (M_1 M_2) = \Delta (M_1) \Delta (M_2)$$

$$\Delta \left(M \equiv \left[M_i^{i \in 1..|M|} \right] \right) = \left[\Delta \left(M_i^{i \in \left\{ i \mid M_i \neq \overline{M_j} \text{ for } j \in 1..|M| \right\}} \right) \right]$$

$$(4.13)$$

The key is in the case (4.14) where the argument to Δ is a collection. In this case, the function Δ does two things. First, it removes those pairs of terms in the collection that are opposite. Then, it recursively applies itself to each of the remaining terms.

Note that unlike the observation function Θ of the λ^p -calculus, the observation function Ξ of the λ^q -calculus is not total. For some λ^q -term M, $\Xi(M)$ does not yield a λ -term. An example of such a term is $M \equiv x, \neg x$ because $\Delta(M)$ is the collection M with all pairs of opposites removed. However, the empty collection is not a λ^p -term. Therefore, some λ^q -terms cannot be observed. The non-totality of the observation function Ξ does not limit the λ^q -calculus because careful programming can always insert a unique term into a collection prior to observation to ensure observability. There is thus no need to add distinguished tokens to the λ^q -calculus such as error or unobservable.

Because $\Xi = \Theta \circ \Delta$, the definition of statistical indistinguishability (3.7) applies to $\Xi(M)$ and $\Xi(N)$ as well, if both $\Delta(M)$ and $\Delta(N)$ exist. Although observing a λ^p -term is statistically indistinguishable from observing its γ -normal form, observing a λ^q -term is, in general, statistically distinguishable from observing its γ -normal form.

4.6. Observational Semantics

The observational semantics for the λ^q -calculus is similar to that of the λ^p -calculus (3.20). It is given by a single rule.

$$\frac{M \leadsto v \quad \Xi(v) = N}{M \multimap N} (\text{ObsQ}) \tag{4.15}$$

4.7. Examples

We provide one example. We show how satisfiability may be solved in the λ^q calculus. We assume possible solutions are encoded some way in the λ^q -calculus and there is a term $\underline{\mathrm{CHECK}_f}$ that checks if the fixed Boolean formula f is satisfied by a particular truth assignment, given as the argument. The output from this is a collection of $\underline{\mathrm{T}}$ (truth) and $\underline{\mathrm{F}}$ (falsity) terms. We now present a term that will effectively remove all of the $\underline{\mathrm{F}}$ terms. It is an instance of a more general method.

$$\underline{\text{REMOVE-F}} \equiv \lambda x. \, \underline{\text{IF}} \, x \, x \, (x, \neg x) \tag{4.16}$$

We give an example evaluation.

Observing the final term will always yield $\underline{\mathbf{T}}$. Note that the drawback to this method is that if f is unsatisfiable then the term will be unobservable. Therefore, when we insert a distinguished term into the collection to make it observable, we risk observing that term instead of $\underline{\mathbf{T}}$. At worst, however, we would have a fifty-fifty chance of error.

Specifically, consider what happens when the argument to $\underline{\text{REMOVE-F}}$ is a collection of $\underline{\text{F}}$'s. Then $\underline{\text{REMOVE-F}} = (\underline{\text{F}}, \neg \underline{\text{F}})$. We insert $\underline{\text{I}} \equiv \lambda x.x$ which, if we observe, we take to mean that either f is unsatisfiable or we have bad luck. Thus, we observe the term $(\underline{\text{I}}, \underline{\text{F}}, \neg \underline{\text{F}})$. This will always yield $\underline{\text{I}}$. However, we cannot conclude that f is unsatisfiable because, in the worst case, the term may have been $(\underline{\text{I}}, \underline{\text{REMOVE-F}} \, \underline{\text{T}}) = (\underline{\text{I}}, \underline{\text{T}})$ and we may have observed $\underline{\text{I}}$ even though f was satisfiable. We may recalculate until we are certain to an arbitrary significance that f is not satisfiable.

Therefore, applying REMOVE-F to the results of $\underline{\text{CHECK}_f}$ and then observing the result will yield $\underline{\mathbf{T}}$ only if f is satisfiable.

5. Conclusion

We have seen two new formalisms. The λ^p -calculus allows expression of randomized algorithms. The λ^q -calculus allows expression of quantumized algorithms. In these calculi, observation is made explicit, and terms are presumed to exist in some Heisenberg world of *potentia*.

This work represents a new direction of research. Just as the λ -calculus found many uses, the λ^p -calculus and the λ^q -calculus may help discussion of quantum computation in the following ways.

- 1. Quantum programming languages can be specified in terms of the λ^q -calculus and compared against each other.
- 2. Algorithms can be explored in the λ^q -calculus on a higher level than quantum Turing machines, which, like classical Turing machines, are difficult to program.
- 3. An exploration of the relationship between the λ^q -calculus and quantum Turing machines, quantum computational networks, or other proposed quantum hardware, may provide insights into both fields.

We have seen some algorithms for the λ^p -calculus and the λ^q -calculus. It should not be difficult to see that the λ^p -calculus can simulate a probabilistic Turing machine and that the λ^q -calculus can simulate a quantum Turing machine. It should also follow that a probabilistic Turing machine can simulate the λ^p -calculus, with the exponential slowdown that comes from computing in the world of reality rather than the world of potentia. However, it is not obvious that a quantum Turing machine can simulate the λ^q -calculus. An answer to this question, whether positive or negative, will be interesting. If quantum computers can simulate the λ^q -calculus efficiently, then the λ^q -calculus can be used as a programming language directly. As a byproduct, satisfiability will be efficiently solvable. If quantum computers cannot simulate the λ^q -calculus efficiently, knowing what the barrier is may allow the formulation of another type of computer that can simulate it.

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